QUANTUM GRAVITY CORRECTIONS TO
PARTICLE INTERACTIONS

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An heuristic semiclassical procedure that incorporates quantum gravity induced corrections in the description of photons and spin 1/2 fermions is reviewed. Such modifications are calculated in the framework of loop quantum gravity and they arise from the granular structure of space at short distances. The resulting effective theories are described by power counting non-renormalizable actions which exhibit Lorentz violations at Planck length scale. The modified Maxwell and Dirac equations lead to corrections of the energy momentum relations for the corresponding particle at such scale. An action for the relativistic point particle exhibiting such modified dispersion relations is constructed and the first steps towards the study of a consistent coupling between these effective theories are presented.

1. Introduction

Recently there has been a revival in the interest of studying both observational and theoretical manifestations of quantum gravity induced effects.[1,2,3,4] This means to consider phenomena associated with the Planck scale, which are highly suppressed in standard scenarios. Many of the envisioned effective theories for particles which incorporate Planck scale modifications also induce minute violations of Lorentz covariance at such scale. In this way, these studies naturally overlap with the systematic approach developed by Colladay and Kostelecky [5] which provides the most general power counting renormalizable extension of the standard model that incorporates both Lorentz and CPT violations. This framework has been used to set experimental bounds upon the interactions that produce such violations and the observations performed so far cover a wide range of experimental settings.[6] As pointed out some time ago, the propagation of high energy particles through cosmological distances may provide a realistic possibility to observe one of the effects associated with Planck scale corrections: the modification of the corresponding energy-momentum relations.[2] In the case of photons the authors of Ref.(1) proposed to consider the modified dispersion relations

$$c^2 \, \vec{p}^{\,2} = E^2 (1 + \xi \, E/E_{QG} + O(E/E_{QG})^2),$$

(1)

with E_{QG} being a scale expected to be close to the Planck mass $M_P = 1/\ell_P = 10^{19}\,GeV$. The relation (1) implies an energy dependent photon velocity leading to a time retardation between two photons simultaneously emitted which are detected with a difference in energies ΔE after traveling a distance L. The uncorrected expression for such retardation is

$$\Delta t \quad \approx \quad \xi\,(\Delta E/E_{QG})\,(L/c). \tag{2}$$

In the process of detecting photons from the active galaxy Markarian 421 ($L = 3.5\,l.y.$), Biller et. al.[7] have identified events with $\Delta E = 1\,TeV$ arriving to earth within the time resolution of the measurement: $\Delta t = 280s$. In this way the lower bound $E_{QG}/\xi = 4 \times 10^{16}\,GeV$ is established. Time resolutions of milliseconds have been achieved in recent gamma ray burst (GRB) observations[8] and they will substantially improve up to $10^{-7}s$ with the Gamma Ray Large Area Telescope (GLAST) to be operating in the International Space station by 2006. Nevertheless, the cosmic radiation background (CRB) prevents space to be transparent to the propagation of very high energy photons. For this reason, the detection of ultrahigh energy neutrinos (UHEN) could provide an arena to observe such effects. In fact, the fireball model for the emission of GRB predicts also the generation of $10^{14} - 10^{19}eV$ neutrino bursts. The Extreme Universe Space Observatory (EUSO), already approved for accommodation study on the International Space Station would measure such UHEN, also in coincidence with the photons associated to the burst. As we can see, the near future might provide us with very good chances to detect such quantum gravity effects, or at least to set rather strict bounds on the theories predicting them. As a matter of fact, constraints upon the parameters defining such Lorentz violating theories have already been established by using current observations[9]. The extension of the modified dispersion relations (1) to other particles has provided a way to circumvent some traditional astrophysical paradoxes such as the existence of the GZK cutoff, the observation of multi-TeV photons from Markarian 501 and the pion-stability paradox related to the structure of the air showers produced by high energy cosmic rays.[10] The pioneering work revealing the appearance of Planck scale modifications to photon propagation in the framework of loop quantum gravity [11] was made by Gambini and Pullin.[12] Also in the framework of loop quantum gravity the present author, in a collaboration with J. Alfaro and H. Morales-Técotl, has extended such results by developing an heuristic semiclassical approach which allows the construction of effective field theories for photons and neutrinos, including Planck scale corrections arising as a manifestation of quantum gravitational effects.[13] In particular, dispersion relations of the form (1) arise from such construction. An alternative approach inspired in string theory [14] has been developed by Ellis et. al. and includes both photons and spin 1/2 particles.[15] The paper is organized as follows: in section 2 a brief review of the basic assumptions underlying the semiclassical approximation developed in Ref.(11) is given. The remaining sections contain new material which can be considered as the first steps to investigate a fully consistent coupling of the effective theories involved. Section

3 contains the formulation of the Gambini-Pullin electrodynamics with sources in terms of the standard electromagnetic potentials. In section 4, an action for the relativistic particle yielding Planck scale modified dispersion relations is presented. Finally, in section 5, I discuss the extension to a Dirac particle of the effective action previously found for two-components spin 1/2 fermions.

2. The Effective Theories

Each effective matter Hamiltonian is defined as the expectation value of the corresponding quantum gravity operator in a semiclassical mixed state which describes a flat metric together with the corresponding matter field. The requirements and properties of such a state are made precise in the sequel. The resulting effective theories violate Lorentz covariance at the Planck scale and such violation can be understood as a spontaneous symmetry breaking generated when taking the expectation value in the semiclassical state.

In this section I summarize the procedure for the case of (the magnetic sector of) Maxwell theory. The starting point is the corresponding Hamiltonian

$$H = \int_\Sigma d^3x \, \frac{1}{2} \frac{q_{ab}}{\sqrt{\det q}} [\underline{E}^a \underline{E}^b + \underline{B}^a \underline{B}^b], \tag{3}$$

where the space time is assumed to be a manifold M with topology $\Sigma \times I\!R$. Here Σ is a Riemannian 3-manifold with metric q_{ab}, $\underline{E}^a$ and $\underline{B}^a$ denote the electric and magnetic fields respectively. Thiemann has proposed a general regularization scheme that produces a sound mathematical definition for all the operators entering in the description of loop quantum gravity.[16] Such regularized operators act upon states which are functions of generalized connections defined over graphs. A basis for such space is provided by the so called spin network states. A graph Γ is a set of vertices $v \in V(\Gamma)$ in Σ which are joined by edges e. The regularization procedure is based upon a triangulation of space which is adapted to each graph. This means that the space surrounding any vertex of Γ is filled with tetrahedra Δ having only one vertex in common with the graph (called the basepoint $v(\Delta)$) plus segments $s_I(\Delta)$ starting at Δ and directed along the edges of the graph. In the regions not including the vertices of Γ the choice of tetrahedra is arbitrary and the results are independent of it. The arcs connecting the end points of $s_I(\Delta)$ and $s_J(\Delta)$ are denoted by $a_{IJ}(\Delta)$ and the loop $\alpha_{IJ} := s_I \circ a_{IJ} \circ s_J^{-1}$ can be formed. A fundamental property of this procedure is the use of the volume operator $\hat{V}$ as a convenient regulator. In this way, the action of the operators is finite and gets concentrated only in the vertices of the graph.

In the case of the magnetic sector of (3) Thiemann's regularization leads to the operator

$$\hat{H}^B = \frac{1}{2\,\hbar^2\kappa^2} \sum_{v \in V(\Gamma)} \left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \epsilon^{JKL}\,\epsilon^{MNP} \times$$

$$\times \quad \hat{w}_{i\,L\Delta}\,\hat{w}_{i\,P\Delta'}\,\left(\underline{h}_{\alpha JK(\Delta)}-1\right)\left(\underline{h}_{\alpha MN(\Delta')}-1\right). \tag{4}$$

where

$$\hat{w}_{i\,L\Delta} \quad = \quad tr\left(\tau_i\,h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)},\sqrt{\hat{V}_v}\right]\right) \tag{5}$$

and $E(v)$ is a factor related to the number of edges that joint at the vertex v. In Eqs.(4, 5) $h_{\alpha(\Delta)}$ ($\underline{h}_{\alpha(\Delta)}$) denote parallel transport operators (holonomies) of either the gravitational connection A_{ia} or the electromagnetic connection $\underline{A}_a$, respectively, along the path α associated to the tetrahedron Δ. Such holonomies are $SU(2)$ and $U(1)$ group elements, respectively. The corresponding trajectories have been previously defined.

Next, the quantum state that produces the semiclassical approximation is described. To this end let us consider an ensemble of graphs together with their adapted triangulation (which means a set of segments $\{s_I(\Delta)\}$ for each graph), characterized by some probability distribution $P(\Gamma)$. To each graph Γ one associates a wave function $|\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}\rangle$ which is peaked with respect to the classical electromagnetic field configuration together with a flat gravitational metric and a zero value for the gravitational connection. In other words, the contribution for each operator inside the expectation value can be estimated as

$$\langle\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}|\,...\hat{q}_{ab}...\,|\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}\rangle \quad = \quad \delta_{ab}+O\left(\frac{\ell_P}{\mathcal{L}}\right)$$

$$\langle\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}|\,...\hat{A}_{ia}...\,|\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}\rangle \quad = \quad 0+\frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon}, \tag{6}$$

while the expectation values including the electric and magnetic operators are estimated through their corresponding classical values $\vec{E}$ and $\vec{B}$. Not surprisingly, the semiclassical state specifies both the classical coordinate and the classical momentum for each pair of canonical variables. The scale $\mathcal{L}>>\ell_P$ of the wave function is such that the continuous flat metric approximation is appropriate for distances much larger that $\mathcal{L}$, while the granular structure of spacetime becomes relevant when probing distances smaller that $\mathcal{L}$. Such scale will have a natural realization according to each particular physical situation. In the sequel we set $\Upsilon = 0$ for simplicity and denote the semiclassical state as $|\Gamma,S\rangle = |\Gamma,\mathcal{L},\vec{\underline{E}},\vec{\underline{B}}\rangle$.

In a very schematic way I summarize now the method of calculation. For each graph Γ the effective Hamiltonian is defined as $H_\Gamma = \langle\Gamma,S|\hat{H}_\Gamma|\Gamma,S\rangle$. For a given vextex, inside the expectation value, one expands each operator in powers of the segments $s_I(\Delta)$ plus derivatives of the matter operators. In the case of (4) this produces

$$H^B_\Gamma \quad = \quad \sum_{v\in V(\Gamma)}\sum_{v(\Delta)=v}\langle\Gamma,S|\hat{\underline{E}}_{p_1q_1}(v)...\partial^{a_1}...\hat{\underline{E}}_{pq}(v)\hat{T}_{a_1}...^{pq\,p_1q_1\,\cdots}(v,s(\Delta))|\Gamma,S\rangle.$$

$$\tag{7}$$

where $\hat{T}$ contains gravitational operators together with contributions depending on the segments of the adapted triangulation in the particular graph. Next, space is considered to be divided into boxes, each centered at a given point $\vec{x}$ and with volume $\mathcal{L}^3 \approx d^3 x$. The choice of boxes is the same for all the graphs considered. Each box contains a large number of vertices of the semiclassical state ($\mathcal{L} \gg \ell_P$), but it is considered as infinitesimal in the scale where the space can be regarded as continuous. The sum over the vertices in (4) is subsequently split as the sum over the vertices in each box, plus the sum over boxes. Also, one assumes that the electromagnetic operators are slowly varying within a box ($\mathcal{L} \ll \lambda$, with λ been the photon wavelength), in such a way that for all the vertices inside a given box one can write $\langle \Gamma, S| \dots \hat{\underline{E}}_{ab}(v) \dots |\Gamma, S \rangle = \mu \underline{E}_{ab}(\vec{x})$. Here $\underline{E}_{ab}$ is the classical electromagnetic field at the center of the box and μ is a dimensionless constant which is determined in such a way that the standard classical result in the zeroth order approximation is recovered. Applying the procedure just described to (4) leads to

$$
\begin{aligned}
\mathrm{H}_\Gamma^B &= \sum_{\text{Box}} \underline{F}_{p_1 q_1}(\vec{x}) \dots \left(\partial^{a_1} \dots \underline{F}_{pq}(\vec{x}) \right) \sum_{v \in \text{Box}} \ell_P^3 \times \\
&\qquad \sum_{v(\Delta)=v} \mu^{n+1} \langle \Gamma, S| \frac{1}{\ell_P^3} \hat{T}_{a_1 \dots}{}^{pqp_1 q_1 \cdots}(v, s(\Delta)) |\Gamma, S \rangle.
\end{aligned}
\tag{8}
$$

In the above, $n+1$ is the total number of factors $F_{pq}(\vec{x})$. The expectation value of the gravitational contribution is expected to be a rapidly varying function inside each box. Finally, the effective Hamiltonian is defined as an average over the graphs Γ, i.e. over adapted triangulations : $\mathrm{H}^B = \sum_\Gamma P(\Gamma) \, \mathrm{H}_\Gamma^B$. This effectively amounts to average the expectation values remaining in each box of the sum (8). We call this average $T_{a_1 \dots}{}^{pqp_1 q_1 \cdots}(\vec{x})$ and estimate it by demanding T to be constructed from the flat space tensors δ_{ab} and ϵ_{abc}. In this way one is imposing isotropy and rotational invariance on our final result. Also the scalings given in (6) together with the additional assumptions: $\langle \Gamma, S| ... \hat{V} ... |\Gamma, S \rangle \longrightarrow \ell_P^3, s_I^a \longrightarrow \ell_P$ are used . Let us remark that the above average can be understood as taking the expectation value of the Hamiltonian (4) in a mixed state characterized by the density matrix $\rho = \sum_\Gamma |\Gamma, S\rangle P(\Gamma) \langle \Gamma, S|$. After replacing the summation over boxes by the integral over space, the resulting Hamiltonian has the final form

$$
\mathrm{H}^B = \int d^3 x \, \underline{F}_{p_1 q_1}(\vec{x}) \dots \left(\partial^{a_1} \dots \underline{F}_{pq}(\vec{x}) \right) \, T_{a_1 \dots}{}^{pqp_1 q_1 \cdots}(\vec{x}).
\tag{9}
$$

Some comments are now in order. A rigorous semiclassical treatment of loop quantum gravity is still in the process of development.[17] Since the approach presented here has made use only of the main features that semiclassical states should have, all dimensionless coefficients in the expectation values that contribute to $T_{a_1 \dots}{}^{pqp_1 q_1 \cdots}(\vec{x})$ in (9) remain undetermined. Besides, the calculation has not been performed in a covariant way. On the contrary, the results are expected to be valid

only in a preferred reference frame. Thus, the states considered so far will not annhilate the Hamiltonian constraint of quantum gravity.

By applying the above procedure to photons (γ) and to two-component massive neutrinos with definite chirality (ν) the following results, including corrections to order ℓ_P^2, are obtained [13]

$$
\mathrm{H}^\gamma = \int d^3\vec{x} \left[\left(1 + \theta_7 \left(\frac{\ell_P}{\mathcal{L}} \right)^2 \right) \frac{1}{2} \left(\underline{\vec{B}}^2 + \underline{\vec{E}}^2 \right) + \theta_8 \ell_P \left(\underline{\vec{B}} \cdot (\vec{\nabla} \times \underline{\vec{B}}) + \underline{\vec{E}} \cdot (\vec{\nabla} \times \underline{\vec{E}}) \right) \right.
$$
$$
\left. + \theta_3 \, \ell_P^2 \left(\underline{B}^a \, \nabla^2 \underline{B}_a + \underline{E}^a \, \nabla^2 \underline{E}_a \right) + \theta_2 \, \ell_P^2 \, \underline{E}^a \partial_a \partial_b \underline{E}^b + \theta_4 \, \mathcal{L}^2 \, \ell_P^2 \, \left(\underline{\vec{B}}^2 \right)^2 + \ldots \right].
$$
$$(10)$$

$$
\begin{aligned}
\mathrm{H}^\nu &= \int d^3 x \left[i \, \pi(\vec{x}) \left(1 + \kappa_1 \frac{\ell_P}{\mathcal{L}} + \ldots - \kappa_2 \, \ell_P^2 \, \nabla^2 \right) \tau^d \partial_d \xi(\vec{x}) + \right. \\
&+ \frac{i}{\mathcal{L}} \, \pi(\vec{x}) \left(\kappa_3 \frac{\ell_P}{\mathcal{L}} + \ldots - \kappa_4 \, \ell_P^2 \nabla^2 \right) \xi(\vec{x}) \\
&+ \left. \frac{m}{2} \xi^T(\vec{x}) \, (i\sigma^2) \left(1 + \kappa_5 \frac{\ell_P}{\mathcal{L}} + \kappa_6 \, \ell_P \, \tau^a \partial_a \right) \xi(\vec{x}) + c.c. \right].
\end{aligned}
$$
$$(11)$$

Here $\pi(\vec{x}) = i\xi^*(\vec{x})$ and $\tau^a = -(i/2) \, \sigma^a$ with σ^a being the standard Pauli matrices. In particular, the effective theories given in (10) and (11) imply Lorentz violating Planck scale modifications of the corresponding particle energy-momentum relations which are calculated in Ref.(11). In both cases the scale $\mathcal{L}$ has been estimated as the De Broglie wavelength of the corresponding particle. In the case of neutrinos the condition $m_\nu << p_\nu$ was assumed.

Before closing this section let us emphasize that the effective theories (10) and (11) are expected to be valid in a particular reference frame, the most natural one being that in which the CRB spectrum looks isotropic. This means that the involved scales ℓ_P and $\mathcal{L}$ will experience FitzGelrald-Lorentz contraction in going to the laboratory frame, for example. Also, the velocity of light will not have a universal value, exhibiting corrections depending on ℓ_P which arise from the modified dispersion relations. An alternative point of view allowing for deformed dispersion relations which are valid in every reference frame has been recently proposed[19] and further elaborated.[20] This requires the formulation of a relativity principle having two observer independent scales: the speed of light constant and the Planck-length constant, which can be realized via non-linear realizations of the Lorentz group. An analysis of the common features and main differences between the approaches of Refs.(19) and (20) has also appeared.[21]

3. The Gambini-Pullin Electrodynamics with Sources

In this section I present the first steps towards a more detailed discussion of the modified electrodynamics obtained in (10). In order to have the correct normaliza-

tion in the zeroth order case ($\ell_P = 0$) it is convenient to make the field redefinition

$$(1 + \theta_7 (\ell_P/\mathcal{L})^2)^{1/2} \underline{E}_i \quad \longrightarrow \quad E_i \tag{12}$$

and similarly for $\underline{B}_i \longrightarrow B_i$. Considering only the contribution linear in ℓ_P one is left with the effective Hamiltonian density

$$\mathcal{H}^{EM} = \frac{1}{2} \left(\vec{B}^2 + \vec{E}^2 \right) + \theta_8 \ell_P \left(\vec{B} \cdot (\vec{\nabla} \times \vec{B}) + \vec{E} \cdot (\vec{\nabla} \times \vec{E}) \right), \tag{13}$$

which was previously obtained by Gambini and Pullin.[12] This theory predicts birefringence effects which are manifest through different propagation velocities for left and right polarized photons. Such velocity difference is proportional to the parameter θ_8 in Eq. (13) and it is linear in ℓ_P. By analizing the presence of linear polarization in the optical and ultraviolet spectrum of some cosmological sources the bound $\theta_8 < 10^{-3}$ is obtained.[18]

Adding the appropriate sources in the first order action

$$S[\Phi, A_i, E_j] = \int dt\, d^3\vec{x} \left(-E_i\, \dot{A}_i - \mathcal{H}^{EM} + \Phi\, (\partial_i\, E_i - 4\pi\, \rho) + 4\pi J_i A_i \right), \tag{14}$$

with $\vec{B} = \nabla \times \vec{A}$ and the potential Φ acting as a Lagrange multiplier, the modified Maxwell equations

$$\nabla \cdot \vec{E} = 4\pi\rho, \quad \nabla \cdot \vec{B}, \;\; = \;\; 0$$

$$\vec{\nabla} \times \left(\vec{B} + 2\theta_8\ell_P \nabla \times \vec{B} \right) - \frac{\partial \vec{E}}{\partial t} \;\; = \;\; 4\pi\vec{J},$$

$$\vec{\nabla} \times \left(\vec{E} + 2\theta_8\ell_P \nabla \times \vec{E} \right) + \frac{\partial \vec{B}}{\partial t} \;\; = \;\; 0, \tag{15}$$

are obtained. From Eqs. (15) one can prove the continuity equation for the electric charge as a signal of consistency. In order to write a second order Lagrangian formulation of Maxwell equations (15) in terms of the basic fields $A^\mu = (\Phi, \vec{A})$ it is convenient to reintroduce the potentials starting, as usual, from the homogeneous equations. Here I use the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the conventions of Jackson. The magnetic field retains the standard relation with the vector potential: $B_i = \epsilon_{ijk} F_{jk}$, while the relation defining the electric field is changed to

$$\vec{E} + 2\theta_8\ell_P \nabla \times \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = F_{0i}. \tag{16}$$

Eq. (16) implies that the resulting Lagrangian will be non local. In order to invert (16), the operator

$$M_{ij}^{-1}(x, y) = \delta^4(x - y)\, (\delta_{ij} + 2\theta_8\ell_P\epsilon_{ikj}\partial_k) = \delta(x^0 - y^0) M_{ij}^{-1}(\vec{x}, \vec{y}), \tag{17}$$

is defined in such a way that

$$E_i(x) = \int d^4y\, M_{ij}(x, y) F_{0j}(y). \tag{18}$$

The Lagrangian density is $\mathcal{L} = -E^i \dot{A}_i - \mathcal{H}$, where the velocities $\dot{A}_i$ are introduced via the equation (18), which defines E_i as a non local function of the potentials. This leads to

$$\mathcal{L} = \frac{1}{2} \left(\vec{E}^2 - \vec{B}^2 \right) + \theta_8 \ell_P \left(\vec{E} \cdot (\vec{\nabla} \times \vec{E}) - \vec{B} \cdot (\vec{\nabla} \times \vec{B}) \right) - 4\pi J^\mu A_\mu. \qquad (19)$$

The above Lagrangian density violates P, but preserves C and T. The CPT violation is produced because (19) is both non local and non invariant under proper Lorentz transformations. Nevertheless, (19) is invariant under rotations and thus it is valid in a preferred coordinate system which we identify with that in which the CRB looks isotropic. It is interesting to observe that the Lagrangian density (19) is power counting non-renormalizable, thus constituting a case which is not considered in the general framework of Ref.(3) to discuss Lorentz and CPT violations. All effective theories generated via the framework presented here are expected to enjoy such non-renormalizability property. Also, (19) describes an effective theory which should be valid only at energy scales much lower than the Planck mass. A convenient way of presenting the action arising from (19) is

$$S = \int d^4x d^4y \, \frac{1}{2} \left(E_i(x) \, M_{iq}^{-1}(x,y) \, E_q(y) - B_i(x) \, M_{iq}^{-1}(x,y) \, B_q(y) \right) - 4\pi \int d^4x J^\mu A_\mu$$

$$(20)$$

where the operator $M_{iq}^{-1}(x,y)$ satisfies the additional symmetry condition $M_{iq}^{-1}(x,y) = M_{qi}^{-1}(y,x)$. Varying the action (20) leads indeed to the modified Maxwell equations (15). Before closing this section I discuss the explicit inversion of the operator $M_{iq}^{-1}(\vec{x}, \vec{y})$. In momentum space, with the convention $\nabla = i\,\vec{k}$, one has

$$\left(M^{-1} \right)_{im} = \delta_{im} + (i\kappa\ell_P)\, \epsilon_{ilm}\, k_l, \quad \varsigma = i\kappa\ell_P, \qquad (21)$$

which inverse is

$$M_{ij} = \frac{1}{1 + \varsigma^2 \vec{k}^2} \left(\delta_{ij} + \varsigma\epsilon_{ijp} k_p + \varsigma^2 \, k_i k_j \right). \qquad (22)$$

The pole in (22) signals the need of a cutoff in order to regulate the Fourier transform. This is in accordance with the effective character of the theory, which is no more valid for momenta close to the pole position $k_\infty = 1/(\kappa\ell_P)$. In coordinate representation, (22) leads to the Fourier transform

$$M(\vec{z}) = \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \frac{1}{1 - \kappa^2\ell_P^2 \vec{k}^2} = \frac{4\pi}{z} \int k dk \, \frac{\sin kz}{1 - (\kappa^2\ell_P^2)\, k^2}. \qquad (23)$$

For energies which are low compared to k_∞, it will be enough to consider the local approximation to the operator $M_{ij}(\vec{x}, \vec{y})$, which in turn leads to a simpler local effective theory. This is done by expanding the denominator of (23) in power series of ℓ_P and integrating. The result, up to second order in ℓ_P is

$$M_{ij}(\vec{z}) = (2\pi)^3 \, \delta^3(\vec{z}) \left(\delta_{ij} + \kappa\ell_P\epsilon_{ijp}\partial_p + \kappa^2\ell_P^2 \left(\partial_i\partial_j - \delta_{ij}\nabla^2 \right) + O(\kappa^3\ell_P^3) \right), \qquad (24)$$

with $\vec{z} = \vec{x} - \vec{y}$.

4. An Action for the Relativistic Particle

In order to consistently couple the electrodynamics considered in section 3 to a point particle it is necessary to have an action for the latter which naturally incorporates corrections to the dispersion relations of the type discussed previously. A simple way to do this is by starting from a first order action where the corresponding modified energy-momentum relation is incorporated as a constraint. In other words, suppose that one needs to consider the modified dispersion relation

$$F(p^0) - (\vec{p}^2 + m_0^2) = 0. \tag{25}$$

The action to consider is

$$S = \int d\tau \left(p_0 \, \dot{x}^0 - p^i \, \dot{x}^i - \frac{\lambda}{2} \left(F(p^0) - (\vec{p}^2 + m_0^2) \right) \right), \tag{26}$$

where λ is a Lagrange multiplier. The equations of motion are

$$\delta p_0 \quad : \quad \dot{x}^0 = \frac{\lambda}{2} \frac{dF}{dp^0} \Longrightarrow \frac{2\dot{x}^0}{\lambda} = G(p^0) \Longrightarrow p^0 = G^{-1} \left(\frac{2\dot{x}^0}{\lambda} \right),$$

$$\delta p^i \quad : \quad -\dot{x}_i + \lambda p^i = 0 \Longrightarrow p^i = \frac{\dot{x}^i}{\lambda}, \tag{27}$$

together with (25). Here $G(u) = dF(u)/du$. Substituting p^0 and p^i back into the action yields

$$S \quad = \quad \int d\tau \left(\dot{x}^0 G^{-1} \left(\frac{2\dot{x}^0}{\lambda} \right) - \frac{\lambda}{2} F \left(G^{-1} \left(\frac{2\dot{x}^0}{\lambda} \right) \right) - \frac{\dot{x}^i \dot{x}^i}{2\lambda} + \frac{\lambda}{2} m_0^2 \right). \tag{28}$$

The second order action is obtained by eliminating the auxiliary field λ via its equation of motion arising from (28). In order to illustrate the procedure in a more tractable situation I choose

$$F(p^0) \quad = \quad \left(p^0 \right)^2 + \alpha \ell_P \frac{1}{3} \left(p^0 \right)^3, \qquad G(p^0) = 2p^0 + \alpha \ell_P p^{0^2} = y,$$

$$G^{-1}(y) \quad = \quad \frac{1}{\alpha \ell_P} \left(\sqrt{(1 + \alpha \ell_P y)} - 1 \right). \tag{29}$$

In the approximation linear in ℓ_P I obtain

$$G^{-1}(y) = \frac{1}{2} y - \frac{1}{8} \alpha \ell_P y^2, \quad F(G^{-1}(y)) = \frac{1}{4} y^2 - \frac{1}{12} \alpha \ell_P y^3. \tag{30}$$

Clearly, the dispersion relation arising from the choice (29) together with (25) reproduces Eq.(1) in the zero mass limit. Substituting in (28), and after some algebra, one obtains

$$S \quad = \quad \int d\tau \left[\frac{v^2}{2\lambda} - \frac{1}{6} \alpha \ell_P \frac{\left(\dot{x}^0 \right)^3}{\lambda^2} + \frac{\lambda}{2} m_0^2 \right], \quad v^2 = \left(\dot{x}^0 \right)^2 - \left(\dot{x}^i \right)^2. \tag{31}$$

The equation of motion for λ is

$$-\frac{v^2}{\lambda^2} + \frac{2}{3}\alpha\ell_P\frac{\left(\dot{x}^0\right)^3}{\lambda^3} + m_0^2 \;\; = \;\; 0, \tag{32}$$

which is also solved to first order in ℓ_P by making the ansatz $\lambda = \lambda_0\left(1 + \ell_P\lambda_1\right)$. The result is

$$\lambda_0 = \frac{\sqrt{v^2}}{m_0}, \qquad \lambda_1 = -\frac{1}{3}\alpha\frac{m_0\left(\dot{x}^0\right)^3}{\left(v^2\right)^{3/2}}, \tag{33}$$

leading to the action

$$S = m_0\int dt\,\left[\sqrt{1-\vec{v}^2} - \frac{\alpha}{6}\left(m_0\ell_P\right)\frac{1}{\left(1-\vec{v}^2\right)}\right]. \tag{34}$$

Observe that the limit $\ell_P = 0$ correctly reproduces the well known relativistic action for the point particle. From the previous equations one can write the energy and momentum in terms of the velocity as

$$p^0 = m_0\gamma - \frac{\alpha}{2}\ell_P m_0^2\gamma^2\left(1 - \frac{2}{3}\gamma^2\right), \quad p^i = m_0\gamma v^i\left(1 + \frac{\alpha}{3}\left(m_0\ell_P\right)\gamma^3\right). \tag{35}$$

At this stage it is appropriate to compare the results (35) with those of Ref.(20), which, to first order in ℓ_P, are

$$p^0 = m_0\gamma - \ell_P m_0^2\gamma^2, \qquad p^i = m_0\gamma v^i\left(1 - \ell_P m_0\gamma\right). \tag{36}$$

The discrepancy is telling us that the action (34) is not an scalar under the specific non-linear representation of the Lorentz group proposed in Ref.(20). Finally, it is interesting to emphasize that Planck scale corrections to either particle propagation or quantum field interactions need not necessarily imply violations of Lorentz covariance.[19,20,21,22]

5. A Lagrangian for Dirac Particles

The aim in this section is to construct a modified Lagrangian density for Dirac particles, starting from the theory given by (11). Such two component theory can be embedded into a four component realization by demanding Ψ to be a Majorana spinor

$$\Psi = \begin{bmatrix} \xi \\ \chi \end{bmatrix} = \begin{bmatrix} \xi \\ -i\,\sigma^2\xi^* \end{bmatrix}. \tag{37}$$

In this notation, the equations arising from (11) are

$$\left[i\frac{\partial}{\partial t} - i\hat{A}\,\vec{\sigma}\cdot\nabla\right]\xi - m\left(1 - iC\,\ell_P\,\vec{\sigma}\cdot\nabla\right)\chi = 0,$$

$$\left[i\frac{\partial}{\partial t} + i\hat{A}\,\vec{\sigma}\cdot\nabla\right]\chi - m\left(1 - iC\,\ell_P\,\vec{\sigma}\cdot\nabla\right)\xi = 0, \tag{38}$$

with $\hat{A} = \left(1 + D\,\ell_P^2\,\nabla^2\right)$ and C, D being constants. For simplicity I discuss here the rather unrealistic case $\mathcal{L} \to \infty$. Next, χ and ξ are considered to be independent spinors. In the conventions where γ_5 is diagonal, $(\gamma_0)^2 = 1$ and the signature is $(+ - - -)$, one can verify that

$$\left(i\gamma^\mu \partial_\mu + i\left(D\,\ell_P^2\,\nabla^2\right)\,\vec{\gamma}\cdot\nabla - m\left(1 - iC\ell_P\,\vec{\Sigma}\cdot\nabla\right)\right)\Psi = 0 \qquad (39)$$

reproduces the equations (38). The spin operator is given by $\Sigma^k = (i/2)\epsilon_{klm}\gamma^l\gamma^m$. The Lagrangian density that yields Eq. (39) is

$$\mathcal{L}_D = \frac{1}{2}\left(\,i\bar{\Psi}\gamma^\mu\left(\partial_\mu\Psi\right) + iD\,\ell_P^2\left(\nabla^2\bar{\Psi}\right)\gamma^k\left(\partial_k\Psi\right) - m\bar{\Psi}\left(1 - iC\ell_P\Sigma^k\,\partial_k\right)\Psi + h.c.\right),$$
$$(40)$$

which is invariant under the global phase transformation $\delta\Psi = i\delta\Theta\Psi$. The associated Noether current, which will be identified with the electromagnetic current is

$$J^0 = \bar{\Psi}\gamma^0\Psi, \qquad J^k = \bar{\Psi}(\gamma^k + C\,m\ell_P\,\Sigma^k)\Psi, \qquad (41)$$

to first order in ℓ_P. The coupling with the modified electrodynamics (19) is made in the standard gauge invariant way via the replacement $\partial_\mu \to \partial_\mu + ieA_\mu$ in (40).

Acknowledgments

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